established by Kossel between the frequencies of the K and L lines hold true for the tungsten lines.

A more detailed account of these and other experiments on the tungsten spectrum, including the distribution of energy in the continuous spectrum, will be published shortly in the *Physical Review*.

[1] Mosely, *Phil. Mag.*, **26**, 210 and 1024 (1913); **27**, 710 (1914).
[2] I. Malmer, *Phil. Mag.*, **28**, 787 (1914).
[3] W. H. Bragg, *Phil. Mag.*, **29**, 407 (1915).
[4] Duane & Hunt, *Physic. Rev.*, **6**, 166 (1915).
[5] Barnes, *Phil. Mag.*, **30**, 368 (1915).
[6] Rutherford, Barnes and Richardson, *Phil. Mag.*, **30**, 339 (1915).
[7] Hull, *Physic. Rev.*, **7**, 156 (1916).
[8] Gorton, *Physic. Rev.*, **7**, 203 (1916).
[9] Kossel, *Ber. D. Physik. Ges.* **16**, 953 (1914).
[10] Webster, these Proceedings, **2**, 90 (1916).

ON THE FOUNDATIONS OF PLANE ANALYSIS SITUS

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The notions point, line, plane, order, and congruence are fundamental in Euclidean geometry. Point, line and order (on a line) are fundamental in descriptive geometry. Point, limit-point and regions (of certain types) are fundamental in analysis situs. It seems desirable that each of these doctrines should be founded on (developed from) a set of postulates (axioms) concerning notions that are fundamental for that particular doctrine. Euclidean geometry and descriptive geometry have been so developed.[1] The present paper contains two systems of axioms, Σ_2 and Σ_3, each of which is sufficient for a considerable body of theorems in the domain of plane analysis situs. The axioms of each system are stated in terms of a class, S, of elements called *points* and a class of sub-classes of S called *regions*.

On the basis of Σ_2, *the existence of simple continuous arcs*[2] *is proved as a theorem.*

The system Σ_2 contains an axiom (Axiom 1) which postulates the existence of a countable sequence of regions containing a set of subsequences that close down in a specified way on the points of space. Among other things this axiom implies that the set of all points is separable.[3]

The system Σ_3 is obtained from Σ_2 by replacing Axioms 1, 2, and 4 by three other axioms, Axioms 1′, 2′, and 4′ respectively. Here Axiom 1′ postulates the existence, *for each point P*, of a countable sequence of regions

that closes down on P, Axiom $2'$ postulates that every two points of a region are the extremities of at least one simple continuous arc that lies in that region, and Axiom $4'$ postulates that every region plus its boundary possesses the Heine-Borel property.

An open curve is defined as a closed, connected[4] set of points K such that if P is any point of K then $K-P$ is the sum of two mutually exclusive connected point-sets neither of which contains a limit point[4] of the other one. It is proved that every open curve is unbounded[4] and divides the set of all remaining points into two domains.

Though Σ_3 is a sufficient basis for a very considerable part of plane analysis situs, nevertheless there exist spaces that satisfy Σ_3 but are neither metrical, descriptive[5] nor separable.

It is interesting that no space that satisfies Σ_3 can be potentially descriptive without being also separable and potentially metrical. Indeed if to Σ_3 there be added the axiom that there exists a system of open curves such that through every two points there is one and only one curve of this system, the resulting set of axioms is categorical with respect to *point* and *limit point of a point-set.*

Every space that satisfies Σ_2 satisfies also Σ_3, but not conversely. In every space satisfying Σ_2 there exists infinitely many open curves through any two given points. I have not however determined whether every such space is descriptive.

Definitions.—A point P is said to be a *limit point* of a point-set M if, and only if, every region that contains P contains at least one point of M distinct from P. The *boundary* of a point-set M is the set of all points $[X]$ such that every region that contains X contains at least one point of M and at least one point that does not belong to M. If M is a set of points, M' denotes the point-set composed of M plus its boundary. A set of points K is said to be *bounded* if there exists a finite set of regions $R_1, R_2, R_3 \ldots R_n$ such that K is a sub-set of $(R_1+R_2+R_3+\ldots+R_n)'$. If R is a region the point-set $S-R'$ is called the *exterior* of R. A set of points is said to be *connected* if however it be divided into two mutually exclusive sub-sets, one of them contains a limit point of the other one.

The System Σ_2.—Axiom 1. *There exists an infinite sequence of regions,* $K_1, K_2, K_3, \ldots$ such that (1) if m is an integer and P is a point, there exists an integer n greater than m, such that K_n contains P, (2) if P and $\bar{P}$ are distinct points of a region R, then there exists an integer δ *such that if* $n>\delta$ *and* K_n *contains* P *then* K_n' *is a subset of* $R-\bar{P}$.

Axiom 2. *Every region is a connected set of points.*

Axiom 3. *If* R *is a region,* $S-R'$ *is a connected set of points.*

Axiom 4.[6] *Every infinite set of points lying in a region has at least one limit point.*

Axiom 5. *There exists an infinite set of points that has no limit point.*

Axiom 6.′ *If R is a region and AB is an arc such that $AB-A$ is a subset of R then $(R+A)-AB$ is a connected set of points.*

Axiom 7′. *Every boundary point of a region is a limit point of the exterior of that region.*

Axiom 8. *Every simple closed curve is the boundary of at least one region.*

The System Σ_3.—The system Σ_3 is composed of Axioms 1,′ 2,′ 3, 4′, 5, 6,′ 7,′ and 8, where Axioms 1′, 2′, and 4′ are as follows:

Axiom 1′. *If P is a point, there exists an infinite sequence of regions, R_1, R_2, R_3, . . . such that* (1) *P is the only point they have in common,* (2) *for every n, R_{n+1} is a proper subset of R_n,* (3) *if R is a region containing P then there exists n such that R_n'* is a subset of R.

Axiom 2′. *If A and B are two distinct points of a region R then there exists, in R, at least one simple continuous arc from A to B.*

Axion 4′. *If R is a region, R' possesses the Heine-Borel property.*

An example of a system satisfying Σ_2 is obtained if in ordinary Euclidean space of two dimensions, the term region is applied to every bounded connected set of points M, of connected exterior, such that every point of M is in the interior of some triangle that lies wholly in M.

Details (including a third system of axioms, the system Σ_1) will appear in *Transactions of the American Mathematical Society*, probably in April, 1916.

[1] Cf., among others, D. Hilbert, *The Foundations of Geometry* (translation by E. J. Townsend, Chicago, 1902) and O. Veblen, A system of axioms for geometry, *Trans. Amer. Math. Soc.*, **5**, 343 (1904).

[2] Lennes defines a continuous simple arc connecting two points A and $B, A \neq B$, as a bounded, closed, connected set of points $[A]$ containing A and B such that no proper connected subset of $[A]$ contains A and B. Cf. N. J. Lennes, *Amer. J. Math.*, **33**, 308 (1911).

[3] A set of points M is said to be *separable* if it contains a countable subset K such that every point of M is a limit point of K. See M. Fréchet, Sur quelques points du calcul fonctionnel, *Palermo, Rend. Circ. Mat.*, **22**, 6 (1906).

[4] For definitions of these terms see below.

[5] A space S is said to be *descriptive*, or *potentially descriptive*, if it contains a system of open curves such that through every two points of S there is one and only one curve of this system.

[6] In view of a result due to F. Hausdorff, it is clear that in the presence of the other axioms of Σ_2, Axiom 4 is equivalent to the axiom that if R is a region then R' possesses the Heine-Borel property. See F. Hausdorff, Grundzüge der Mengenlehre, Veit and Comp., Leipzig, 1914.